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TRANSFER OF RESONANCE RADIATION IN A HALF SPACE

Yu. Yu. Abramov, A. M. Dykhne and A. P. Napartovich*

ABSTRACT. The Wiener-Hopf method is used to solve the problem of the transfer of resonance radiation in the approximation assuming complete frequency mixing. Nonresonance processes such as absorption of photons and damping of excited atoms are also considered. The method suggested allows known results plus a number of new results to be produced in the same manner. Asymptotic expressions are produced in explicit form for excited atom density with arbitrary line form. The influence of frequency redistribution on the behavior of the solutions is discussed. Where possible, a physical interpretation of the results produced is given.

Introduction. The solution of the transfer equations in consideration of space and time heterogeneity is an extremely complex problem. Therefore, in order to produce any results capable of interpretation, various approximations must be used. /459[1]

Even when stationary problems are analyzed, various models of the medium are usually introduced [1], for example it is assumed that the medium is homogeneous and has only two division boundaries with a vacuum (plane layer). In this case, the result is generally determined by the behavior of the functions near the boundary [1-3]. Clearly, with sufficiently thick layers we can limit ourselves generally to an analysis of a half space (one division boundary). However, even in this case the problem of finding precise solutions encounters a number of difficulties. If it is assumed that the scattering centers are nonmobile, and that the scattering is elastic, the transfer equation becomes an integral equation with a difference kernel (Milne equation). In analyzing the problem of diffusion of resonance radiation, this equation can be produced on the assumption that the frequency of light is retained in each scattering event (coherent scattering). In this case we find that the diffusion approximation is valid far from the division boundary. /460 However, the assumption of complete coherence of scattering may be fulfilled only if the atoms of the medium are non-mobile, and the line has only its natural length. Since in practice these conditions are usually not realized, the opposite assumption can be made, that scattering is completely noncoherent, i.e. that the probability of emission by an atom of a quantum of a certain frequency is independent of the frequency of the quantum absorbed.

[1] Numbers in the margin indicate pagination in the foreign text.

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Biberman [4] and Holstein [5] showed for the first time that the diffusion approximation is inapplicable to noncoherent scattering of light, since it is impossible in this case to introduce the concept of the mean free path length. These authors reduced the transfer equation for the case of completely noncoherent scattering (which will be analyzed throughout the following) to integral equations for the concentration of excited atoms. If we know this concentration, we can easily calculate the intensity of radiation at any point in space.

Subsequently, the transfer equation was analyzed in many works [2, 3, 6-12].

In recent time, the theory of radiation transfer in a line has been advanced considerably in the works of Ivanov [2, 3, 6, 9, 10, 13].

In particular, he showed a method of finding the resolvent of the integral equation for a medium of infinitely great optical thickness, and he also analyzed the problem of luminance of the medium, when there are no sources at finite distances (the Milne problem). A number of numerical calculations of practical value in astrophysics have been performed for Gaussian line form. Damping of excited atoms has been introduced as a nonresonance process.

In this work, we analyze the problem of stationary transfer of resonance radiation in a half space.

In order to solve the equation, we use the Viener-Hopf method. A number of results produced earlier by Sobolev [1] and Ivanov [2, 3, 6, 9, 10, 13] are concluded in the same manner, using a Laplace transform. The method suggested allows us to consider absorption in a continuous spectrum. In addition to the general asymptotics for the density of excited atoms $n(x)$, we shall find expressions for the density of outgoing radiation $J_\nu(0,\mu)$ at frequencies far from the resonant frequency. A convenient integral representation is presented for the basic function usually used in calculations.

1. *Statement of problem. Equations.* Following [1, 4], we will limit ourselves to analysis of a medium consisting of atoms with two levels, which absorb and radiate light in a resonance manner. Obviously, at frequencies near the resonance frequency, we can ignore various nonresonance processes due to their relatively low cross section. /461

It is natural to expect that nonresonance processes leading to finite path lengths of photons at frequencies far from the resonance frequency do not essentially change the spectral distribution at the center of the line or the number of excited atoms at distances not too far from the division boundary.

In this sense, the solution near the center of the line will be universal (independent of the type of predominant nonresonance processes).

In order to trace the influence of nonresonance processes on the course of the solution at the middle area of frequencies and distances, let us analyze the

constant absorption of impurities evenly included in the medium and the damping of excited atoms with constant cross section σ. We consider the absorption cross section $\epsilon \ll k_{\nu_0}$ to be the coefficient of resonance absorption of photons at the center of the line.

The selection of these processes is determined by the fact that the model, as before, allows precise solution.

We note that where $\epsilon = 0$ and $\sigma = 0$, the natural unit of measurement of distance is the free path length at the center of the line. If ϵ is not 0, the characteristic distance is also the path length of a photon resulting from absorption by an impurity (in this case ϵ^{-1}). It is natural to expect that the influence of nonresonance processes will not be great at frequencies where $k_\nu \gg \epsilon$, or at distances from the boundary $x \ll \epsilon^{-1}$. In this case, the asymptotics of the solution where $x \gg 1$ will be essentially different in areas $x \ll \epsilon^{-1}$ and $x \gg \epsilon^{-1}$.

In the case where $\sigma \neq 0$, the problem is somewhat more complex, since the value of σ is the cross section of damping of excited atoms, the concentration of which is determined by the density of radiation at this point. The change of density of radiation with distance depends essentially on the concrete form of the absorption coefficient; therefore, with various forms of function k_ν, the areas of change of the coordinates for which the various asymptotics are correct will change somewhat [12].

The areas of applicability of the asymptotics should depend on the rate with which $k_\nu \to 0$ where $|\nu - \nu_0| \to \infty$. With a sharper drop in function k_ν, the influence of σ will be felt earlier.

For convenience, throughout the following we will assume all values to be expressed in dimensionless units. The distance should be measured in path lengths at the center of the line. The coefficient of absorption at the center of the line will be taken as 1, and frequency will be measured in line widths and counted from resonance frequency ν_0. The coefficient of absorption is /462
proportional to the coefficient of radiation; therefore, they can be considered equal to each other. This makes the phase density of radiation $J_\nu(x,\mu)$ also dimensionless. The radiation transfer equation has the form [1]:

$$\mu \frac{\partial J_\nu}{\partial x} = -k_\nu J_\nu(x, \mu) + \frac{k_\nu}{4\pi} n - \varepsilon J_\nu(x, \mu) + \frac{f_\nu(x, \mu)}{4\pi}, \qquad (1)$$

where arc cos μ is the angle between the direction of propagation of the radiation at a given point and the x axis, k_ν is the absorption coefficient,

n is the concentration of atoms in the excited state, measured in relative units, $f_\nu(x,\mu)$ is a function used to consider radiation from secondary sources.

The density of excited atoms is determined by the equation:

$$2\pi\int_{-\infty}^{\infty} k_\nu d\nu \int_{-1}^{1} J_\nu(x,\ \mu)\, d\mu - n\left(\int_{-\infty}^{\infty} k_\nu d\nu + \sigma\right) = 0 \tag{2}$$

(σ is the damping cross section). It is assumed that the temperature is independent of the coordinates. Also, since the rate of movement of the atoms $v \ll c$, equation (2) contains no term proportional to grad n.

System (1)-(2) allows us to produce easily a single integral equation for the density of excited atoms, having the following form [1]:

$$n(x) = \int_0^\infty K(x-t)\, n(t)\, dt + F(x), \tag{3}$$

where

$$K(x-t) = \frac{1}{2(\bar{k}+\sigma)} \int_{-\infty}^{\infty} d\nu k_\nu^2 \int_0^1 \frac{e^{-(k_\nu+\sigma)\frac{|x-t|}{\mu}}}{\mu}\, d\mu,\quad \bar{k} = \int_{-\infty}^{\infty} k_\nu d\nu. \tag{4}$$

Similarly, function F(x) is expressed through integrals of the power of the sources $f_\nu(x,\mu)$; we will not present the corresponding equation, as it is somewhat cumbersome.

2. *Solution of principal equations.* For the resolvent of integral equation (3), we have:

$$\Gamma(x,\ x') = \int_0^\infty K(x-t)\,\Gamma(t,\ x')\, dt + K(x-x'). \tag{5}$$

Since the kernel of the equation is symmetrical,

$$\Gamma(x, x') = \Gamma(x', x).$$

We note that the equation for the resolvent corresponds to the equation for the concentration of excited atoms in the case where a "standard" source of photons is located at point x'; in this case /463

$$f_*(x, \mu) = k_*\delta(x - x') \text{ и } F(x) = K(x - x')). \tag{6}$$

Using the properties of integral operator K, it is not difficult to produce [14]:

$$\Gamma(x, x') = \Gamma(0, |x - x'|) + \int_0^{\min(x, x')} \Gamma(0, x - u)\,\Gamma(0, x' - u)\,du. \tag{7}$$

Thus, in order to solve equation (5), it is sufficient to find the function $\Gamma(0,x)$. According to (5)

$$\Gamma(0, x) = \int_0^{\infty} K(x - t)\,\Gamma(0, t)\,dt + K(x). \tag{8}$$

Equation (8) can be solved using the Wiener-Hopf method. For our purposes, it is convenient to use a two-sided Laplace transform. Using ordinary symbols,

$$f^+(p) = \int_0^{\infty} e^{-px} f(x)\,dx, \qquad f^-(p) = \int_{-\infty}^{0} e^{-px} f(x)\,dx,$$

and defining function $\Gamma(0,x)$ where $x < 0$ through the relationship

$$\Gamma(0, x) = \int_0^{\infty} K(x - t)\,\Gamma(0, t)\,dt + K(x), \quad x < 0,$$

we produce from (8):

$$\Gamma^+(p)+\Gamma^-(p)=K(p)\Gamma^+(p)+K(p).$$

It is not difficult to see that the "plus" functions in this last equation are analytic in the half plane Re $p > 0$, while the "minus" functions are analytic in the half plane Re $p < \epsilon$.

Representing $G(p) \equiv 1 - K(p)$, we can represent it in the form

$$G(p)=\frac{G^+(p)}{G^-(p)}$$

where /464

$$\ln G^+(p)=\frac{-1}{2\pi i}\int_{-\gamma-i\infty}^{-\gamma+i\infty}\frac{\ln G(z)\,dz}{z-p}, \qquad \operatorname{Re} p>-\gamma. \tag{9}$$

The integral in (9) is taken along a straight line in the analytic area of function G(p), and where Re $p > -\gamma$, it determines a function which is analytic in the right half plane. Since γ may be arbitrarily close to $k\epsilon$, function (9) will be analytic in the area Re $p > -\epsilon$.

Where Re $p < -\epsilon$, we will take $G_+^+(p)$ to mean a function produced as an analytic extension of the function $G_+^-(p)$ defined with Re $p > -\epsilon$.

We can similarly define function $G^-(p)$, analytic in half plane Re $p < \epsilon$.

$$\ln G^-(p)=\frac{-1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\ln G(z)\,dz}{z-p}, \qquad \operatorname{Re} p<\gamma. \tag{10}$$

Formulas (9) and (10) can be used if function G(p) does not vanish in the analytic band $-\epsilon < p < \epsilon$.

Let us prove that where ϵ and σ are other than 0 $G(p) \neq 0$ in the entire plane of the complex variable. Following [1], let us convert formula (4) to a more convenient form using the replacement $u = (k_\nu + \epsilon)/\mu$ and a substitution in the order of integration. In this case we produce

$$K(x) = \frac{1}{\bar{k} + \sigma} \int_0^\infty Q(u)\, e^{-(u+\varepsilon)|x|}\, du, \tag{11}$$

where

$$Q(u) = \frac{1}{u + \varepsilon} \int_{\nu(u)}^\infty \bar{k}_\nu^2 d\nu, \tag{12}$$

and the function $\nu(u)$ is determined as follows:

$$k_\nu = u \Big| \text{where} \qquad \text{and} \qquad \text{where } u \geqslant 1. \tag{13}$$

Consequently, the Laplace transform of the kernel is:

$$K(p) = \int_{-\infty}^\infty e^{-px} K(x)\, dx = \frac{2}{\bar{k} + \sigma} \int_0^\infty \frac{(\varepsilon + u)\, Q(u)\, du}{(\varepsilon + u)^2 - p^2}, \tag{14}$$

from which it follows directly that the function K(p) is analytic in the plane with cross sections on the real axis where $|\text{Re } p| > \epsilon$. Where $\epsilon = 0$, the cross section passes along the entire real axis and formula (14) defines two functions which are analytic in nonintersecting areas. /465

Since $G(p) \equiv 1 - K(p)$, in order to prove that $G(p) \neq 0$, we must be assured that the equation $K(p) = 1$ has no solution. It follows from (14) that K(p) is real only with imaginary p and on the sector of the real axis from $-\epsilon$ to ϵ, and that throughout this area $K(p) < K(\epsilon)$.

In this case

$$K(\varepsilon) = \frac{1}{\bar{k} + \sigma} \left[\int_0^\infty \frac{Q(u)\, du}{u} + \int_0^\infty \frac{Q(u)\, du}{u + 2\varepsilon} \right].$$

From this we can see that $K(\epsilon)$ decreases with increasing ϵ. Where $\epsilon = 0$, the integrals are easily selected and we produce

$$K(0) = \frac{\bar{k}}{\bar{k} + \sigma} < 1,$$

which proves our affirmation.

As we can see from the proof, the case $\epsilon = 0$ is a special case. We will investigate it, causing ϵ to approach 0 in the final formulas.

The integral equation for the function $\Gamma(0,x)$, written in Laplace transforms, will be:

$$\Gamma^+(p)\frac{G^+(p)}{G^-(p)} + \Gamma^-(p) = 1 - \frac{G^+(p)}{G^-(p)},$$

or

$$\Gamma^+(p)\,G^+(p) + G^+(p) = -\Gamma^-(p)\,G^-(p) + G^-(p). \tag{15}$$

The left side of the equality contains functions which are analytic in the area $\mathrm{Re}\ p > 0$, while the right portion contains functions analytic in the area $\mathrm{Re}\ p < \epsilon$.

Thus, expression (15) can be only a polynomial of p, but since $\Gamma^+(p) \to 0$, $G^+(p) \to 1$, but this polynomial is identical to 1. From this we find:
$p\to\infty$ $p\to\infty$

$$\Gamma^+(p) = \frac{1}{G^+(p)} - 1, \tag{16}$$

and

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} e^{px}\Gamma^+(p)\,dp = \begin{cases} \Gamma(0,x) & \text{where } x>0, \\ 0 & \text{where } x<0, \end{cases} \tag{17}$$

δ is an arbitrary positive constant.

Let us now transform expression (9), defining function $G^+(p)$, deforming the integration contour to the cross sections in the left half plane. The integral /466

with respect to large circumferences disappears in this case, since G(p) → 1 as p → ∞, and we have

$$\ln G^+(p) = -\frac{1}{2\pi i}\int_{\Gamma}^{\infty} \frac{\ln G(z-i\delta) - \ln G(z+i\delta)}{z+p}\,dz, \quad \delta \to 0. \tag{18}$$

According to (14)

$$G(z-i\delta) = 1 - \frac{1}{\bar{k}+\sigma}\int_0^{\infty}\left[\frac{Q(u)}{\varepsilon+u-z+i\delta} + \frac{Q(u)}{\varepsilon+u+z-i\delta}\right]du =$$

$$= 1 - \frac{2}{\bar{k}+\sigma}\int_0^{\infty} \frac{(\varepsilon+u)\,Q(u)\,du}{(\varepsilon+u)^2 - z^2} + \frac{i\pi Q(z-\varepsilon)}{\bar{k}+\sigma}$$

and similarly

$$G(z+i\delta) = 1 - \frac{2}{\bar{k}+\sigma}\int_0^{\infty} \frac{(\varepsilon+u)\,Q(u)\,du}{(\varepsilon+u)^2 - z^2} - \frac{i\pi Q(z-\varepsilon)}{\bar{k}+\sigma}.$$

Let us substitute this into (18) and, using the fact that

$$\ln\frac{1+iz}{1-iz} = 2i \operatorname{arc\,tg} z.$$

Then

$$\ln G^+(p) = -\frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{arc\,tg}\dfrac{\pi Q(z)}{(\bar{k}+\sigma)\,\Phi(z)}}{\varepsilon+z+p}\,dz, \tag{19}$$

where

$$\Phi(z) = 1 - \frac{2}{k + \sigma_0} \int_0^\infty \frac{(\varepsilon + u)\, Q(u)\, du}{(\varepsilon + u)^2 - (z + \varepsilon)^2}. \tag{20}$$

Thus converting in (17) the integral which defines $\Gamma(0,x)$ where $x > 0$, we produce, similar to [7],

$$\Gamma(0, x) = \int_0^\infty \frac{e^{-\varepsilon x - px}\, Q(p)\, G^+(p + \varepsilon)}{\Phi^2(p) + Q^2(p)}\, dp. \tag{21}$$

The convenience of the expression which we have produced (19) for function /467
$G^+(p)$ is the fact that with simultaneous calculation of the intensity of radiation and the concentration of atoms, we use the same functions.

3. *Particular cases. Relationship to other methods.* When the resolvent is found, the problem of the transfer of resonance radiation in a half space is in principle solved; however, frequently, in order to produce concrete results, it is more convenient to use the Wiener-Hopf method, applying it directly to the integral equation for the density of excited atoms (3).

Using this method, it is not difficult to produce an expression for the Laplace transform of function n(x) with arbitrary sources, i.e. the quantity $n^+(p)$. $n^+(p)$ is used to express the intensity of the radiation leaving the medium [1]:

$$J_\nu^{(sc)}(\mu) = \frac{k_\nu}{4\pi |\mu|}\, n^+\left(\frac{k_\nu + \varepsilon}{|\mu|}\right). \tag{22}$$

In particular, as was noted above, for $f_\nu(x,\mu) = k_\nu \delta(x)$ (flat "standard" source at the coordinate origin), function n(x) corresponds to $\Gamma(0,x)$ and, consequently, (16) and (22),

$$J_\nu(0, \mu) = \frac{k_\nu}{4\pi |\mu|}\left[\frac{1}{G^+\left(\frac{k_\nu + \varepsilon}{|\mu|}\right)} - 1\right]. \tag{23}$$

Let us analyze also the case when a monochromatic stream of frequency ν_0 (not to be confused with the resonance frequency, which is equal to zero in our units) strikes the medium from without at an angle of arc cos μ_0 to the

surface ($f_\nu(x,\mu) = \delta(x)\delta(\nu - \nu_0)\delta(\mu - \mu_0)$). The integral equation for the density of excited atoms in this case will have the form

$$n(x) = \int_0^\infty K(x-t)\, n(t)\, dt + \frac{k_{\nu_0}}{2\mu_0} \frac{e^{-\frac{k_{\nu_0}+\varepsilon}{\mu_0}x}}{\bar{k}+\sigma}. \tag{24}$$

Using the Wiener-Hopf method, it is simple to produce, similar to (16):

$$n^+(p) = \frac{k_{\nu_0}}{2\mu_0} \frac{1}{(\bar{k}+\sigma)} \left[\left(p + \frac{k_{\nu_0}+\varepsilon}{\mu_0}\right) G^+(p)\, G^+\left(\frac{k_{\nu_0}+\varepsilon}{\mu_0}\right)\right]^{-1} \tag{25}$$

and, according to (22), the intensity of the exiting radiation will be:

$$J_\nu(0,\ \mu) = \frac{k_\nu \cdot k_{\nu_0}}{8\pi(\bar{k}+\sigma)|\mu||\mu_0|} \times$$

$$\times \frac{1}{\left(\frac{k_\nu+\varepsilon}{|\mu|} + \frac{k_{\nu_0}+\varepsilon}{\mu_0}\right) G^+\left(\frac{k_{\nu_0}+\varepsilon}{\mu_0}\right) G^+\left(\frac{k_\nu+\varepsilon}{\mu}\right)}. \tag{26}$$

/468

These formulas were produced by another method in [1].

Function n(x), which is the solution of equation (24), depends in this case on the frequency (ν_0) of the incident light, the incidence angle (μ_0) and the coordinates. As a function of these quantities, it corresponds with an accuracy to a factor with the probability of exit of a quantum from the medium, as touched upon in works [1, 2, 6]. Knowing the solution of equation (24), it is easy to find the Laplace transform of function n(x) for arbitrary sources, and consequently, according to (22), the intensity of the exiting radiation.

Let us represent by $n_q(x)$ a function which is the solution of the equation

$$n_q(x) = \int_0^\infty K(x-t)\, n_q(t)\, dt + e^{-qx}.$$

According to (24) and (25),

$$n_q^+(p) = \frac{1}{(p+q)\,G^+(p)\,G^+(q)} \tag{27}$$

and

$$n_q(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{xz}dz}{(z+q)\,G^+(z)\,G^+(q)}.$$

On the other hand,

$$n_q(x) = \int_0^\infty \Gamma(x,\ x')\,e^{-qx'}\,dx' + e^{-qx} = \Gamma^+(x,\ q) + e^{-qx}, \tag{28}$$

where $\Gamma(x,x')$ is the resolvent of the integral equation introduced above, $\Gamma^+(x,q)$ represents the transform of function $\Gamma(x,x')$ with respect to the variable x'. In the general case, the solution of integral equation (3) is expressed through the resolvent as follows (considering the symmetry of function $\Gamma(x,x')$)

$$n(x) = \int_0^\infty \Gamma(x',\ x)\,F(x')\,dx' + F(x).$$

Performing in this equation a Laplace transform with respect to variable x and using (28), we find /469

$$n^+(p) = \int_0^\infty n_p(x')\,F(x')\,dx'$$

and according to (25):

$$J_{\nu}^{(sc)}(p) = \frac{k_{\nu}}{4\pi|\mu|}\int_0^{\infty} n^{\frac{k_{\nu+\varepsilon}}{|\mu|}}(x)F(x)dx.$$

Function $G^{+}(p)$ is encountered in almost all calculation formulas. We note that it corresponds with the function $1/[H(1/p)]$. The function H(z) has been used in many works [1, 2, 6, 7, 9, 11]. Where $\epsilon = 0$, the integral in (9) can be selected with respect to the imaginary axis. Going over in it by a simple replacement to the real variable and using the parity of function G(z), the integral can be taken to the limits $(0,\infty)$. The formula thus produced corresponds to the formula in [6]. Usually, function H(z) is determined in the works cited by the equation:

$$H(z) = 1 + \frac{1}{k+\sigma} zH(z)\int_0^{\infty} \frac{H(z')}{z+z'} \frac{Q\left(\frac{1}{z'}\right)}{z'} dz'.$$

With our approach, it can be easily produced, using the linearity of the transfer equation and applying the principle of superposition.

Let us analyze the solution of the transfer equation (1) in the second case :

$$f_{\nu}^{(1)} = k_{\nu}\delta(x) \text{and} f_{\nu}^{(2)} = \delta(x)\,\delta(\mu - \mu_0)\,\delta(\nu - \nu_0).$$

Obviously,

$$f_{\nu}^{(1)} = \int_{-\infty}^{\infty} d\nu_0 \int_0^1 d\mu_0 k_{\nu_0} f_{\nu}^{(2)}.$$

In this case, due to the linearity,

$$n^{(1)+}(p) = \int_{-\infty}^{\infty} d\nu_0 \int_0^1 d\mu_0 k_{\nu_0} n^{(2)+}(p), \tag{29}$$

where $n^{(2)+}(p)$ depends both on the parameters of μ_0 and ν_0.

For the first case, as was produced, $n^{(1)+}(p) = \Gamma^{+}(p) = 1/[G^{+}(p)] - 1$. For /470
the second case $n^{(2)+}(p)$ is given by formula (25).

Substituting these expressions in (29), after some transformations, we produce the equation presented above for the function H(p).

4. *Milne problem.* We should make particular note of those cases when sources at finite distances are not present. The corresponding problem is called the Milne problem; it arises in analysis of the transmission of light through a plate of very great optical thickness. Like [2, 6], we will produce the solution in this case by placing a source at finite distance x_0 and then causing x_0 to approach infinity. Obviously, in this case we should increase the intensity of the source in the corresponding manner, so that the limit of the solution remains finite. The solution thus produced does not depend on the type of source and, generally speaking, satisfies a heterogeneous integral equation.

The simplest transition to the problem without sources is produced if we assume at the start that at distance x_0 there is a "standard" (6) source of intensity $A(x_0)$. As a heterogeneous term in the integral equation, in this case we use $A(x_0)K(x - x_0)$; its solution where $x < x_0$, as follows from (7), is:

$$A(x_0)\,\Gamma(x,\ x_0) = A(x_0)\,\Gamma(0,\ x_0 - x) + A(x_0)\int_0^x \Gamma(0, x-u)\,\Gamma(0, x_0-u)\,du.$$

Let us select function $A(x_0)$ such that $\lim_{x_0\to\infty} A(x_0)(0, x_0 - x)$ is finite.
According to (21), $\Gamma(0,x)$ as $x \to \infty$ acts basically as $\exp(-\epsilon_x)$.

Therefore, assuming $A(x_0) = (c/\Gamma(0,x_0))$, we produce

$$\lim_{x_0\to\infty} c\,\frac{\Gamma(0,\ x_0 - x)}{\Gamma(0,\ x_0)} = ce^{sx}.$$

The density of excited atoms

$$\psi(x) \equiv \lim_{x_0\to\infty} \frac{c\Gamma(x,\ x_0)}{\Gamma(0,\ x_0)}$$

in this case will be:

$$\psi(x) = ce^{\varepsilon x}\left(1 + \int_0^x e^{\varepsilon(u-x)}\,\Gamma(0,\ x-u)\,du\right) = ce^{\varepsilon x}\left(1 + \int_0^x e^{-\varepsilon t}\,\Gamma(0,\ t)\,dt\right), \qquad (30)$$

/471

from which it follows that $c = \psi(0)$.

It is not difficult to see that the function $\psi(x)$ satisfies the integral equation

$$\psi(x) = \int_0^\infty K(x-t)\,\psi(t)\,dt + \psi(0)\,G^+(-\varepsilon)\,e^{\varepsilon x} \qquad (31)$$

and that

$$\psi^+(p) = \frac{\psi(0)}{(p-\varepsilon)\,G^+(p)}. \qquad (32)$$

We note that where $\sigma = \epsilon = 0$, equation (31) becomes homogeneous. We will arbitrarily call the problem with a source at infinity homogeneous, although the equation for the concentration of excited atoms (31) is, generally speaking, heterogeneous. As follows from (31) and the properties of function $G^+(p)$, this equation becomes homogeneous only where $\sigma = \epsilon = 0$.

5. *Asymptotic behavior of function $G^+(p)$.* Since the function $G^+(p)$ is included in almost all calculation formulas, it is of interest to analyze its behavior with small and large p. Let us analyze first the case $p \gg 1$. We divide the integral in (19) which determines the function $G^+(p)$ into two parts: $\int_0^1$ and $\int_1^\infty$. Since p is great, in the first integral we can perform our expansion with respect to the relationship $z/p \ll 1$, and the order of its decrease as $p \to \infty$ is $1/p$. The second integral, where $p \gg 1$, is equal to $-\frac{\bar{k}^2}{2(\bar{k}+\sigma)}\frac{\ln p}{p} + \frac{c'}{p} + 0\left(\frac{1}{p}\right)$.

Thus, where $p \gg 1$ [1, 2]

$$\ln G^+(p) = 1 - \frac{\bar{k}^2}{2(\bar{k}+\sigma)}\frac{\ln p}{p} + \frac{c'}{p} + 0\left(\frac{1}{p}\right), \qquad (33)$$

where $\overline{k^2} = \int_{-\infty}^{\infty} k_\nu^2 d\nu$, c' – is a constant on the order of 1.

We note that the asymptotic behavior of function $G^+(p)$ as $p \to \infty$ does not depend on the form of the absorption coefficient (in formula (33) we find quantities $\overline{k}$ and $\overline{k^2}$, which characterize only quite roughly the properties of the medium). The asymptotics of (33) is therefore universal and does not change even in the case when the redistribution by frequencies is completely absent. /472

The case of small p is somewhat more complex. In order to analyze the properties of $G^+(p)$ at small p, we note that it follows from formulas (9) and (1)), as well as from the parity of function G(p), that:

$$G^+(p) = 1/G^-(-p)$$

and

$$G(p) = \frac{G^+(p)}{G^-(p)} = G^+(p)\,G^+(-p). \tag{34}$$

Let us investigate the behavior of function G(p) at zero. We use for G(p) the expression

$$G(p) = 1 - \frac{1}{2(\overline{k} + \sigma)p}\int_{-\infty}^{\infty} k_\nu^2 \ln\frac{k_\nu + \varepsilon + p}{k_\nu + \varepsilon - p}\,d\nu, \tag{35}$$

which can be easily produced from (14), by changing the order of integration with respect to u and ν. Then

$$[pG(p)]' = -\frac{1}{2(\overline{k} + \sigma)}\left[\int_{-\infty}^{\infty}\frac{k_\nu^2 d\nu}{k_\nu + \varepsilon + p} + \int_{-\infty}^{\infty}\frac{k_\nu^2 d\nu}{k_\nu + \varepsilon - p}\right] + 1. \tag{36}$$

Let us analyze the integral

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{k_\nu^2 d\nu}{k_\nu + t} = -\int_0^1 \frac{u^2\nu'(u)\,du}{u + t}.$$

Here $\nu(u)$ is the function inverse to k_ν, $\nu'(u) = d\nu/du$.

Integrating further by parts, we produce:

$$\int_{-\infty}^{\infty} \frac{k_\nu^2 d\nu}{k_\nu + t} = \bar{k} - 2\int_0^1 \frac{t^2\nu(u)\,du}{(t+u)^2}.$$

By replacing s = u/t, we convert the integral in the right portion to the following form:

$$\nu(t)\, t\int_0^{1/t} \frac{\nu(st)\,ds}{\nu(t)(1+s)^2}. \qquad /473$$

Since $\lim_{t\to 0} \frac{\nu(ts)}{\nu(t)} = \phi(s)$ is finite with any finite s (which follows from the fact that $\nu(t)$ is integrable at 0 and increases monotonously as $t \to 0$), the main term in the last expression with small t is

$$t\nu(t)\int_0^{\infty} \frac{\varphi(s)\,ds}{(1+s)^2}.$$

We note that where $p, \epsilon \ll 1$,

$$\int_0^p (\varepsilon+t)\nu(\varepsilon+t)\,dt \simeq \gamma(\varepsilon+p)^2\nu(\varepsilon+p) - \gamma\varepsilon^2\nu(\varepsilon),$$

where

$$\gamma = \lim_{t\to 0} \frac{\int_0^p t\nu(t)\,dt}{t^2\nu(t)} = \lim_{t\to 0} \frac{t\nu(t)}{2t\nu(t) + t^2\nu'(t)} = \frac{1}{2 + \lim_{t\to 0} \frac{t\nu'(t)}{\nu(t)}}.$$

Then, integration with respect to p produces from (36):

$$G(p) \simeq \frac{\sigma}{\bar{k}+\sigma} + \frac{\alpha\gamma}{(\bar{k}+\sigma)p}\left[(\varepsilon+p)^2\nu(\varepsilon+p) - (\varepsilon-p)^2\nu(\varepsilon-p)\right]. \quad (37)$$

Here

$$\alpha = \int_0^\infty \frac{\varphi(s)\,ds}{(1+s)^2}.$$

This expression for G(p) was produced with $p < \epsilon$ and has right analytic behavior where $p, \epsilon \ll 1$, since $\nu(p)$ for small p is definite and real with positive p and has its cross section in the complex plane which should be drawn with respect to the negative portion of the real axis. Then, analytic continuation of (37) gives us G(p) with all complex $|p| \ll 1$.

After substituting (37) into (18), we produce, by analogy with the conclusion of formula (37):

$$G^+(p) \simeq \sqrt{\frac{\sigma}{\bar{k}+\sigma}}\left(1+\frac{\alpha\gamma}{\sigma p}\left[(\varepsilon+p)^2\nu(\varepsilon+p) - \varepsilon^2\nu(\varepsilon)\right]\right), (p, \varepsilon \to 0). \quad (38)$$

Formula (38) is correct under the condition:

$$(\varepsilon+p)^2\nu(\varepsilon+p) \ll \sigma p \quad (39)$$ /474

and with arbitrary relationship between ϵ and p.

In particular, where $\epsilon = 0$

$$G^+(p) \simeq \sqrt{\frac{\sigma}{\bar{k}+\sigma}}\left(1+\frac{\alpha\gamma}{\sigma}p\nu(p)\right). \quad (40)$$

On the strength of (39), expression (38) is inapplicable where $\sigma = 0$.

However, when $\epsilon = \sigma = 0$, the expression for function $G^+(p)$ as $p \to 0$ can be produced as follows. We note that if $p = 0$ is not an essentially singular point of function f(p), as $p \to 0$ $|f(p)| \approx |f(pe^{i\phi})|$, i.e. the modulus of the function approaches the limit value regardless of the sequence of points selected. Since $G^+(p)$ is positive where $p > 0$,

$$|G^+(pe^{i\varphi})| \simeq G^+(p), \quad p > 0.$$

Then, it follows from equation (34) with positive p that

$$G^+(p) \simeq \sqrt{|G(p)|} \simeq \sqrt{G(ip)}, \quad p \to 0. \tag{41}$$

Where $\epsilon = 0$, formula (37) becomes inconvenient due to its too complex form; therefore, we analyze this case individually. Where $\epsilon = 0$, (36) can be converted:

$$[pG(p)]' = 1 + \frac{2}{\bar{k}+\sigma}\int_0^1 \frac{u^3\nu'(u)\,du}{u^2-p^2}.$$

Using the identity $\frac{u^3}{u^2-p^2} \equiv u + \frac{up^2}{u^2-p^2}$ and proceeding as in the conclusion of formula (37), we produce:

$$G(ip) \simeq \frac{\sigma}{\bar{k}+\sigma} - \frac{2\gamma\beta}{\bar{k}+\sigma}p^2\nu'(p), \qquad \text{where} \qquad \beta = \int_0^\infty \frac{\varphi(s)\,ds}{1+s^2}.$$

Substituting this expression where $\sigma = 0$ in (41), we find finally where $p \to 0$ and $\epsilon = \sigma = 0$

$$G^+(p) \simeq p\sqrt{\frac{2\gamma\beta|\nu'(p)|}{\bar{k}}}. \tag{42}$$

6. *Summary of asymptotics. Discussion.* The approximate expressions produced in the preceding section for functions $G^+(p)$ allow us to determine the intensity of radiation leaving the medium at small angles to the surface or at frequencies far from the resonant frequency. /475

Also, using formula (21), we can find the behavior of the density of excited atoms at small and large distances from the surface. In particular, it follows from (21) that where $x \to \infty$ and ϵ or $\sigma \neq 0$, the asymptotics of function $\Gamma(0,x)$ corresponds to the asymptotics of the kernel of integral equation $K(x)$, namely

$$\Gamma(0, x) \simeq \frac{G^+(\varepsilon)(\bar{k}+\sigma)^2}{[\sigma + 2\alpha\gamma\varepsilon\nu(\varepsilon)]^2}K(x), \quad x \to \infty; \ \varepsilon \text{ or } \sigma \neq 0. \tag{43}$$

Where $\epsilon = 0$, we produce from this:

$$\Gamma(0, x) \simeq \sigma^{-3/2}(k+\sigma)^{+3/2} K(x). \tag{44}$$

The value of K(x) is proportional to the concentration of atoms at point x, excited by radiation arriving without scattering, when a standard light source is located at the coordinate origin. $\Gamma(0,x)$ is the complete number of atoms at point x excited both by direct and by scattered radiation. The identical behavior of these functions as $x \to \infty$ indicates that in the presence of nonresonance processes, the intensity of scattered radiation decreases at great distances in the same manner as radiation passing through the medium without scattering, which results from the great role of long path photons in this area.

The correspondence of the asymptotics of functions $\Gamma(0,x)$ and K(x) where $\epsilon = 0$ was noted by Ivanov [9].

When $\epsilon = \sigma = 0$, this correspondence of asymptotics does not occur. Physically this is related to the fact that now at large distances the main role is played by photons with a frequency near the resonance frequency. Nonresonance processes, which might lead to disappearance at long distances are now absent.

From (6) it is not difficult to see that where $|x - x'| \gg 1$ and $x \gg 1$

$$\Gamma(x, x') \simeq \Gamma(0, |x - x'|).$$

At long distances from the sources and surface of the medium, the behavior of function n(x) is identical for all types of sources. This result follows from the assumption of complete frequency redistribution. A quantum, being absorbed, completely "forgets" the type of its source. However, these asymptotics depend essentially on the form of the absorption line as $\nu \to \infty$ and, therefore, make up a chaaracteristic of the material. Table 1 shows expressions /476
for the functions $\Gamma(0,x)$ and $\psi(x)$ at long distances from the surface.

These asymptotics follow from the expressions for $G^+(p)$ where $p \to 0$ found in the preceding section and from formula (21). We note that the value of p is in this case a small parameter with respect to which expansion is performed. Also, there are two more small parameters, ϵ and σ, considering the influence of nonresonance processes.

The simplest asymptotics is produced when there is a definite relationship between these parameters.

Function $g(\sigma)$ is determined by the relationship $\sigma = g\nu(g)$, ν is the inverse function of k_ν, $c_{1,2,3}$ are positive constants.

TABLE 1

k_ν		$k_\nu = f(\nu)$	$k_\nu = \frac{1}{1+\nu^2}$	$k_\nu = e^{-\nu^2}$
$x \gg \frac{1}{\varepsilon}$	$\psi(x)$ ∞	$e^{\varepsilon x}$	$e^{\varepsilon x}$	$e^{\varepsilon x}$
	$\Gamma(0, x)$ ∞	$x^{-4} e^{-\varepsilon x} \nu'\left(\frac{1}{x}\right)$	$x^{-3/2} e^{-\varepsilon x}$	$x^{-3} (\ln x)^{-1/2} e^{-\varepsilon x}$
$1 \ll x \ll \frac{1}{\varepsilon}, \frac{1}{g(\sigma)}$	$\psi(x)$ ∞	$x\left[\nu'\left(\frac{1}{x}\right)\right]^{-1/2}$	$x^{1/4}$	$x^{1/4} (\ln x)^{1/4}$
	$\Gamma(0, x)$ ∞	$\left[\nu'\left(\frac{1}{x}\right)\right]^{-1/2}$	$x^{-3/4}$	$x^{-1/2} (\ln x)^{1/4}$
$\frac{1}{g(\sigma)} \ll x \ll \frac{1}{\varepsilon}$	$\psi(x)$ ∞	$1 + c_1 x^{-2} \nu'\left(\frac{1}{x}\right)$	$1 - c_2 x^{-1/2}$	$1 - c_3 x^{-1} (\ln x)^{-1/2}$
	$\Gamma(0, x)$ ∞	$x^{-3} \nu'\left(\frac{1}{x}\right)$	$x^{-3/2}$	$x^{-2} (\ln x)^{-1/2}$

We note that the asymptotics presented in the table for the Laurence and Gaussian forms of the lines and $\epsilon = 0$ (actually, $x \ll 1/\epsilon$) were produced by another method in [2, 6]. The new elements here are the first column and the first row of the table.

Using formula (33) for $G^+(p)$, we can conclude that at small x

$$\Gamma(0, x) \simeq a \ln \frac{1}{x} + b + 0(1), \quad x \ll 1. \tag{45}$$

Due to the universality of the asymptotics of (33), formula (45) is correct for any form of function k_ν. The density of excited atoms increases at the surface. The reason for this is the isotropic, flat source (a large number of photons radiated at small angles to the surface arrive at the point in question). /477

For the case of arrival of monochromatic light in a beam from without the medium, the behavior of function n(x) at 0, as follows from (25) and (33), is determined by the expression

$$n(x) \simeq n(0) + cx \ln \frac{1}{x} + 0(x), \tag{46}$$

where c is a positive constant. It follows from (46) that at the coordinate origin n(x) increases ($c > 0$). Since where $x \to \infty$, $n(x) \to 0$, the concentration of excited atoms has a maximum at some distance from the boundary. This occurs due to the fact that as we approach the boundary, an ever greater share of photons moves outward without scattering. This maximum has been experimentally observed.

The intensity of the light leaving the medium is expressed by function $G^+(p)$ using formulas (22), (26) and (32) for the case of diffused passage (homogeneous problem) and diffuse reflection (incidence of light beam from without). Using the approximate expressions for $G^+(p)$ with small and large p, we can find the intensity of the radiation leaving the medium at frequencies far from the resonant frequency or at small angles to the surface. It is interesting to note that in the case of the Laurence form of the coefficient of absorption $k_\nu = 1/(1 + \nu^2)$ and $\epsilon = \sigma = 0$, comparison of calculated data with the approximate expressions for function $G^+(p)$ (formulas (33) and (41)) shows that these approximate expressions are correct with a good degree of accuracy not only when $p \ll 1$ or $p \gg 1$, but also when $p \sim 1$. The following approximate formula is fulfilled with an accuracy of no less than 3.5% throughout the entire range of change of p:

$$G^+(p) = \begin{cases} \sqrt[4]{\frac{2p}{9}}, & 0 \leqslant p \leqslant 1.5 \\ \exp\left(-\frac{1}{4}\frac{\ln p}{p} - \frac{0.303}{p}\right), & 1.5 \leqslant p < \infty. \end{cases} \tag{47}$$

Using (47), we can with good accuracy produce the angular and spectral distributions of photons leaving the medium, since in problems with sources on the surface they depend little on ϵ and σ. The edges of the line in this case will be produced with considerable error, since the influence of σ and ϵ is greater when k_ν is small. /478

If we know the behavior of the concentration of excited atoms, it is not difficult to use formula (1) to find the intensity of radiation within the medium -- $J_\nu(x,\mu)$. It is not difficult to see that with increasing distance from the surface, the dependence of $J_\nu(x,\mu)$ on ν and μ becomes ever weaker. As $x \to \infty$, the dependence on the frequency and angle exists only for those frequencies at which $k_\nu \ll \epsilon$.

For frequencies such that $k_\nu \gg \epsilon$, the distribution of intensity at great distances is spherically symmetrical and independent of frequency, i.e. radiation

occurs in "equilibrium" with the medium.

7. *The role of frequency redistribution.* It is interesting to note the manner in which the transition from a problem with noncoherent scattering to the problem of scattering without redistribution of frequency occurs when there is a constant photon path length and when the diffusion approximation can be used at a certain distance from the boundary. In this latter case for the homogeneous problem and $\epsilon = \sigma = 0$, the concentration of excited atoms increases linearly at great distances [14]. Let us assume in this case function k_ν has the form of a step:

$$k_\nu = \begin{cases} k_0, & |\nu| < 1, \\ 0, & |\nu| > 1 \end{cases}$$

Then from (35) we produce ($\epsilon = \sigma = 0$):

$$G(p) = 1 - \frac{k_0}{2p} \ln \frac{k_0 + p}{k_0 - p} \quad \text{and}$$

$$G(p) \simeq \frac{p^2}{3k_0^2} + 0(p^3), \quad p \to 0.$$

Exactly the same expression for G(p) is produced for the case of scattering without change in frequency; therefore, the solution for these two cases is identical. Here, the properties of the solution at large x are influenced by the area of small p, and the linear increase in concentration in the homogeneous problem is related with the fact that p. 478] with small p. It can be shown that if function k_ν decreases sufficiently sharply as $\nu \to \infty$, we can approximately replace it with a "step." As follows from (35), if we find a frequency ν_1 such that for certain p

$$\frac{p}{k_{\nu_1}} \ll 1 \text{ and } p^2 \int_0^{\nu_1} \frac{d\nu}{k_\nu} \gg \frac{1}{p} \int_{\nu_1}^{\infty} k_\nu^2 d\nu,$$

then in this area of p: /479

$$G(p) \simeq \left(\frac{1}{3} \int_0^{\nu_1} \frac{d\nu}{k_\nu} \right) p^2$$

and, consequently, there is an area of linear increase in function $\psi(x)$. In this area, on the one hand, the influence of the boundary is slight and, on the

other hand, long path photons can still be ignored.

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